

# A NOTE ON VASIU-ZINK WINDOWS

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ABSTRACT. We propose a notion of frames and windows that allows an alternative proof of the Vasiu-Zink classification of  $p$ -divisible groups over ramified complete regular local rings by their Breuil windows.

## INTRODUCTION

Let  $k$  be a perfect field of characteristic  $p \geq 3$ . In [VZ], Vasiu and Zink consider the regular local rings  $\mathfrak{S} = W(k)[[t_1, \dots, t_r, u]]$  and  $R = \mathfrak{S}/E\mathfrak{S}$ , where

$$E = u^e + a_{e-1}u^{e-1} + \dots + a_0$$

such that all  $a_i \in W(k)[[t_1, \dots, t_r]]$  are divisible by  $p$  and  $a_0/p$  is a unit. For  $a \in \mathbb{N}$  they consider also  $\mathfrak{S}_a = \mathfrak{S}/u^{ae}\mathfrak{S}$  and  $R_a = R/p^a R$  and write  $\mathfrak{S}_\infty = \mathfrak{S}$  and  $R_\infty = R$ . They define a category of Breuil windows relative to  $\mathfrak{S}_a \rightarrow R_a$  and a compatible system of functors

$$\kappa_a : (\text{Breuil windows rel. } \mathfrak{S}_a \rightarrow R_a) \rightarrow (\text{Dieudonné displays over } R_a).$$

Here Dieudonné displays are equivalent to  $p$ -divisible groups by [Z2].

**Theorem.** *The functor  $\kappa_a$  is an equivalence for all  $a \in \mathbb{N} \cup \{\infty\}$ .*

For  $a = 1$  this is proved in [Z3], while  $a = \infty$  is the main result of [VZ]. In this note we show that deformations from  $a$  to  $a + 1$  of Breuil windows and of Dieudonné displays are equivalent because both are classified by lifts of the Hodge filtration. With appropriate definitions, the known proof for Dieudonné displays (recalled below) covers both cases. Technically, the main point is to separate the formalism of  $f_1$  from divided power constructions.

By induction it follows that  $\kappa_a$  is an equivalence for finite  $a$ ; the case  $a = \infty$  follows by passing to the projective limit. Since the initial case  $a = 1$  can be shown similarly, this proof is essentially self-contained.

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## 1. FRAMES AND WINDOWS

In this section  $p = 2$  is allowed. The following notions of frames and windows do not coincide with the definitions in [Z3].

**Definition 1.1.** A frame is a quintuple  $\mathcal{F} = (S, I, R, f, f_1)$  where  $S$  is a ring,  $R = S/I$  a quotient ring,  $f$  an endomorphism of  $S$ , and  $f_1 : I \rightarrow S$  an  $f$ -linear homomorphism of  $S$ -modules. We require that  $S$  is local and that  $f_1(I)$  generates  $S$  as an  $S$ -module. (In all examples actually  $1 \in f_1(I)$ .)

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If  $\mathcal{F}$  is a frame, for an  $S$ -module  $M$  we write  $M^{(1)} = S \otimes_{f,S} M$ , and for an  $f$ -linear homomorphism of  $S$ -modules  $g : M \rightarrow N$  we denote  $g^\sharp : M^{(1)} \rightarrow N$  its linearisation,  $g^\sharp(s \otimes m) = sg(m)$ . There is a unique element  $\pi \in S$  such that  $f(a) = \pi f_1(a)$  for  $a \in I$ . Namely, if  $f_1^\sharp(t) = 1$  then  $\pi = f^\sharp(t)$ .

**Definition 1.2.** A window over a frame  $\mathcal{F}$  is a quadruple  $\mathcal{P} = (P, Q, F, F_1)$  where  $P$  is a finite free  $S$ -module,  $Q \subseteq P$  is a submodule,  $F : P \rightarrow P$  and  $F_1 : Q \rightarrow P$  are  $f$ -linear homomorphisms of  $S$ -modules, such that

- (1)  $IP \subseteq Q$  and  $P/Q$  is free over  $R$ ,
- (2)  $F_1(ax) = f_1(a)F(x)$  for  $a \in I$  and  $x \in P$ ,
- (3)  $F_1(Q)$  generates  $P$  as an  $S$ -module.

*Remark 1.3.* Here  $F_1$  determines  $F$ . Indeed, if  $f_1^\sharp(t) = 1$ , for  $x \in P$  we have  $F(x) = F_1^\sharp(tx)$ . In particular  $F(x) = \pi F_1(x)$  when  $x$  lies in  $Q$ .

*Remark 1.4.* If  $(P, Q, F, F_1)$  is a window, there is a decomposition of  $S$ -modules  $P = L \oplus T$  with  $Q = L \oplus IT$ , called normal decomposition, and

$$\Psi : L \oplus T \xrightarrow{F_1 + F} P$$

is an  $f$ -linear isomorphism (which means that  $\Psi^\sharp$  is an isomorphism). Conversely, for given finite free  $S$ -modules  $L$  and  $T$ , the set of window structures on  $(P = L \oplus T, Q = L \oplus IT)$  is bijective to the set of  $f$ -linear isomorphisms  $\Psi$  as above.

## 2. FUNCTORIALITY

Assume that  $\mathcal{F}$  and  $\mathcal{F}'$  are frames and  $u \in S'$  is a unit.

**Definition 2.1.** A  $u$ -morphism of frames  $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$  is a ring homomorphism  $\alpha : S \rightarrow S'$  with  $\alpha(I) \subseteq I'$  such that  $f'\alpha = \alpha f$  and  $f'_1\alpha = u\alpha f_1$ . A morphism of frames is a 1-morphism of frames.

Let  $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$  be a  $u$ -morphism of frames.

**Definition 2.2.** If  $\mathcal{P}$  and  $\mathcal{P}'$  are windows over  $\mathcal{F}$  and  $\mathcal{F}'$ , respectively, an  $\alpha$ -morphism  $g : \mathcal{P} \rightarrow \mathcal{P}'$  is a homomorphism  $g : P \rightarrow P'$  of  $S$ -modules with  $g(Q) \subseteq Q'$  such that  $F'g = gF$  and  $F'_1g = ugF_1$ .

For every window  $\mathcal{P}$  over  $\mathcal{F}$  there is a base change  $\alpha_*\mathcal{P}$  over  $\mathcal{F}'$  with an  $\alpha$ -morphism  $\mathcal{P} \rightarrow \alpha_*\mathcal{P}$  such that  $\text{Hom}_\alpha(\mathcal{P}, \mathcal{P}') = \text{Hom}_{\mathcal{F}'}(\alpha_*\mathcal{P}, \mathcal{P}')$ . Clearly this requirement determines the window  $\alpha_*\mathcal{P}$  uniquely. It can be constructed explicitly as follows: If a normal decomposition  $(L, T, \Psi)$  for  $\mathcal{P}$  is chosen, a normal decomposition for  $\alpha_*\mathcal{P}$  is  $(S' \otimes_S L, S' \otimes_S T, \Psi')$  with  $\Psi'(s' \otimes l) = uf'(s') \otimes \Psi(l)$  and  $\Psi'(s' \otimes t) = f'(s') \otimes \Psi(t)$ .

## 3. CRYSTALLINE MORPHISMS

**Definition 3.1.** A morphism of frames  $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$  is called crystalline if  $\alpha_* : (\text{windows over } \mathcal{F}) \rightarrow (\text{windows over } \mathcal{F}')$  is an equivalence of categories.

**Theorem 3.2.** Suppose a morphism of frames  $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$  induces an isomorphism  $R \cong R'$  and a surjection  $S \rightarrow S'$  with nilpotent kernel  $\mathfrak{a} \subset S$  which has a filtration  $\mathfrak{a} = \mathfrak{a}_0 \supseteq \dots \supseteq \mathfrak{a}_n = 0$  such that  $f(\mathfrak{a}_i) \subseteq \mathfrak{a}_{i+1}$  and  $f_1(\mathfrak{a}_i) \subseteq \mathfrak{a}_i$  and  $f_1$  is elementwise nilpotent on  $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ . Then  $\alpha$  is crystalline.

The proof is a variation of [Z1], Theorem 44 and [Z2], Theorem 3.

*Proof.* Since  $\alpha$  factors as  $\mathcal{F} \rightarrow \mathcal{F}'' \rightarrow \mathcal{F}'$  with  $S'' = S/\mathfrak{a}_1$ , by induction we may assume that  $f(\mathfrak{a}) = 0$  and that  $f_1$  is elementwise nilpotent on  $\mathfrak{a}$ . We may also assume that  $\mathfrak{a}^2 = 0$  because the powers of  $\mathfrak{a}$  are stable under  $f_1$ .

The functor  $\alpha_*$  is essentially surjective since the constituents of a normal decomposition and the  $f$ -linear isomorphism  $\Psi$  can be lifted from  $\mathcal{F}'$  to  $\mathcal{F}$ . In order that  $\alpha_*$  is fully faithful it suffices that  $\alpha_*$  is fully faithful on automorphisms because a homomorphism  $g : \mathcal{P} \rightarrow \mathcal{P}'$  can be encoded by the automorphism  $\begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}$  of  $\mathcal{P} \oplus \mathcal{P}'$ . Since for a window  $\mathcal{P}$  over  $\mathcal{F}$  an automorphism of  $\alpha_* \mathcal{P}$  can be lifted to an  $S$ -module automorphism of  $P$  it suffices to prove the following assertion.

*Assume that  $\mathcal{P} = (P, Q, F, F_1)$  and  $\mathcal{P}' = (P, Q, F', F'_1)$  are two windows over  $\mathcal{F}$  such that  $F \equiv F'$  and  $F_1 \equiv F'_1$  modulo  $\mathfrak{a}$ . Then there is a unique isomorphism  $g : \mathcal{P} \cong \mathcal{P}'$  with  $g \equiv \text{id}$  modulo  $\mathfrak{a}$ .*

We write  $F'_1 = F_1 + \eta$  and  $F' = F + \varepsilon$  and  $g = 1 + \omega$ , where  $\eta : Q \rightarrow \mathfrak{a}P$  and  $\varepsilon : P \rightarrow \mathfrak{a}P$  are given, and  $\omega : P \rightarrow \mathfrak{a}P$  is an arbitrary homomorphism of  $S$ -modules. The induced  $g$  is an isomorphism of windows if and only if  $gF_1 = F'_1g$  on  $Q$ , which translates into

$$(3.1) \quad \eta = \omega F_1 - F_1 \omega - \eta \omega.$$

Here  $\eta \omega = 0$  because for  $a \in \mathfrak{a}$  and  $x \in P$  we have  $\eta(ax) = f_1(a)\varepsilon(x)$ , which is zero as  $\mathfrak{a}^2 = 0$ . We fix a normal decomposition  $P = L \oplus T$  with  $Q = L \oplus IT$ . For  $l \in L$ ,  $t \in T$ , and  $a \in I$  we have

$$\begin{aligned} \eta(l + at) &= \eta(l) + f_1(a)\varepsilon(t), \\ \omega(F_1(l + at)) &= \omega(F_1(l)) + f_1(a)\omega(F(t)), \\ F_1(\omega(l + at)) &= F_1(\omega(l)) + f_1(a)F(\omega(t)). \end{aligned}$$

Here  $F\omega = 0$  because for  $a \in \mathfrak{a}$  and  $x \in P$  we have  $F(ax) = f(a)F(x)$ , and  $f(\mathfrak{a}) = 0$ . Hence (3.1) is equivalent to:

$$(3.2) \quad \begin{cases} \varepsilon = \omega F & \text{on } T, \\ \eta = \omega F_1 - F_1 \omega & \text{on } L. \end{cases}$$

Since  $\Psi : L \oplus T \xrightarrow{F_1+F} P$  is an  $f$ -linear isomorphism, the datum of  $\omega$  is equivalent to the pair of  $f$ -linear homomorphisms

$$\omega_T = \omega F : T \rightarrow \mathfrak{a}P, \quad \omega_L = \omega F_1 : L \rightarrow \mathfrak{a}P.$$

Let  $\lambda : L \rightarrow L^{(1)}$  be the composition  $L \subseteq P \xrightarrow{(\Psi^\#)^{-1}} L^{(1)} \oplus T^{(1)} \xrightarrow{pr_1} L^{(1)}$  and let  $\tau : L \rightarrow T^{(1)}$  be analogous with  $pr_2$  in place of  $pr_1$ . Then (3.2) becomes:

$$(3.3) \quad \begin{cases} \omega_T = \varepsilon|_T, \\ \omega_L - F_1 \omega_L^\# \lambda = \eta|_L + F_1 \omega_T^\# \tau. \end{cases}$$

Let  $U(\omega_L) = F_1 \omega_L^\# \lambda$ . The endomorphism  $F_1$  of  $\mathfrak{a}P$  is elementwise nilpotent because  $F_1(ax) = f_1(a)F(x)$  and because  $f_1$  is elementwise nilpotent on  $\mathfrak{a}$  by assumption. Since  $L$  is finitely generated it follows that  $U$  is elementwise nilpotent, so  $1 - U$  is bijective, and (3.3) has a unique solution.  $\square$

## 4. ABSTRACT DEFORMATION THEORY

The Hodge filtration of a window  $\mathcal{P}$  is the submodule  $Q/IP \subseteq P/IP$ .

**Lemma 4.1.** *Let  $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$  be a morphism of frames such that  $S = S'$ , i.e.  $I \subseteq I'$  is a sub-ideal and  $f'_1$  is an extension of  $f_1$ . Then the base change functor induces an equivalence between the category of windows over  $\mathcal{F}$  and the category of pairs consisting of a window  $\mathcal{P}'$  over  $\mathcal{F}'$  and a lift of its Hodge filtration to a direct summand  $V \subseteq P'/IP'$ .*

*Proof.* Let  $\mathcal{P}'$  over  $\mathcal{F}'$  together with  $V \subseteq P'/IP'$  be given and let  $Q \subset P'$  be the inverse image of  $V$ . In order that  $(P', Q, F', F'_1|_Q)$  is a window we must show that  $F'_1(Q)$  generates  $P'$ . If  $P' = L' \oplus T'$  such that  $Q = L' \oplus IT'$ , this is equivalent to  $F'_1 + F' : L' \oplus T' \rightarrow P'$  being an  $f$ -linear isomorphism, which holds because  $\mathcal{P}'$  is a window.  $\square$

Assume that a morphism of frames  $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$  is given such that  $S \rightarrow S'$  is surjective with nilpotent kernel  $\mathfrak{a}$  and  $I' = IS'$ . We want to factor  $\alpha$  into morphisms of frames

$$(S, I, R, f, f_1) \xrightarrow{\alpha_1} (S, I'', R', f, f'_1) \xrightarrow{\alpha_2} (S', I', R', f', f'_1)$$

such that  $\alpha_2$  satisfies the hypotheses of Theorem 3.2. Necessarily  $I'' = I + \mathfrak{a}$ . The main point is to define  $f'_1 : I'' \rightarrow S$ , which is equivalent to define an  $f$ -linear homomorphism  $f'_1 : \mathfrak{a} \rightarrow \mathfrak{a}$  that extends the restriction of  $f_1$  to  $\mathfrak{a} \cap I$  and satisfies the hypotheses of Theorem 3.2.

If this is achieved, Theorem 3.2 and Lemma 4.1 show that windows over  $\mathcal{F}$  are equivalent to windows  $\mathcal{P}'$  over  $\mathcal{F}'$  together with a lift of the Hodge filtration to a direct summand of  $P/IP$ , where  $\mathcal{P}'' = (P, Q'', F, F'_1)$  is the unique lift of  $\mathcal{P}'$  under  $\alpha_2$ .

## 5. DIEUDONNÉ FRAMES

Let  $R$  be a noetherian complete local ring with maximal ideal  $\mathfrak{m}$  and perfect residue field  $k$  of characteristic  $p$ . If  $p = 2$  we assume that  $pR = 0$ . There is a unique subring  $\mathbb{W}(R) \subset W(R)$  stable under  $f$  such that the projection  $\mathbb{W}(R) \rightarrow W(k)$  is surjective with kernel  $\hat{W}(\mathfrak{m})$ , the ideal of all Witt vectors in  $W(\mathfrak{m})$  whose coefficients converge to zero  $\mathfrak{m}$ -adically. Let  $\mathbb{I}_R$  be the kernel of the natural surjection  $\mathbb{W}(R) \rightarrow R$ .

The Dieudonné frame associated to  $R$  is

$$\mathcal{F}_R = (\mathbb{W}(R), \mathbb{I}_R, R, f, f_1)$$

with  $f_1 = v^{-1}$ ; we note that  $\mathbb{W}(R)$  is a local ring. In this case  $\pi = p$ . Windows over  $\mathcal{F}_R$  are Dieudonné displays over  $R$  in the sense of [Z2]. A local homomorphism  $R \rightarrow R'$  induces a morphism of frames  $\mathcal{F}_R \rightarrow \mathcal{F}_{R'}$ .

Suppose  $R' = R/\mathfrak{b}$  for a nilpotent ideal  $\mathfrak{b}$  equipped with elementwise nilpotent divided powers. The projection  $\mathbb{W}(R) \rightarrow \mathbb{W}(R')$  is surjective with kernel  $\hat{W}(\mathfrak{b}) = W(\mathfrak{b}) \cap \hat{W}(\mathfrak{m})$ . The divided Witt polynomials define an isomorphism of  $\mathbb{W}(R)$ -modules

$$\log : \hat{W}(\mathfrak{b}) \cong \mathfrak{b}^{<\infty>}$$

where  $\mathfrak{b}^{<\infty>}$  is the group of sequences  $[b_0, b_1, \dots]$  with  $b_i \in \mathfrak{b}$  which converge to zero  $\mathfrak{m}$ -adically, and  $x \in \mathbb{W}(S)$  acts by  $[b_0, b_1, \dots] \mapsto [w_0(x)b_0, w_1(x)b_1, \dots]$ . In logarithmic coordinates, the restriction of  $f_1$  to  $\hat{W}(\mathfrak{b}) \cap \mathbb{I}_R$  is given by

$$f_1[0, b_1, b_2, \dots] = [b_1, b_2, \dots].$$

Let  $\tilde{\mathbb{I}} = \mathbb{I}_R + \hat{W}(\mathfrak{b})$ . Then  $f_1$  extends uniquely to an  $f$ -linear homomorphism

$$\tilde{f}_1 : \tilde{\mathbb{I}} \rightarrow \mathbb{W}(R)$$

with  $\tilde{f}_1[b_0, b_1, \dots] = [b_1, b_2, \dots]$  on  $\hat{W}(\mathfrak{b})$ , and we obtain a factorisation

$$(5.1) \quad \mathcal{F}_R \xrightarrow{\alpha_1} \mathcal{F}' = (\mathbb{W}(R), \tilde{\mathbb{I}}, f, \tilde{f}_1) \xrightarrow{\alpha_2} \mathcal{F}_{R'}.$$

The following is a reformulation of [Z2], Theorem 3.

**Proposition 5.1.** *Here  $\alpha_2$  is crystalline.*

It follows that deformations of Dieudonné displays from  $R'$  to  $R$  are classified by lifts of the Hodge filtration; this is [Z2], Theorem 4.

*Proof.* When  $\mathfrak{m}$  is nilpotent,  $\alpha_2$  satisfies the hypotheses of Theorem 3.2; the required filtration of  $\mathfrak{a} = \hat{W}(\mathfrak{b})$  is  $\mathfrak{a}_i = p^i \mathfrak{a}$ . In general, the hypotheses of Theorem 3.2 are not satisfied because  $f_1 : \mathfrak{a} \rightarrow \mathfrak{a}$  is only topologically nilpotent. However, one can find a sequence of ideals  $R \supset I_1 \supset I_2 \dots$  which define the  $\mathfrak{m}$ -adic topology such that each  $\mathfrak{b} \cap I_n$  is stable under the divided powers of  $\mathfrak{b}$ . Then the proposition holds for each  $R/I_n$  in place of  $R$ , and the general case follows by passing to the projective limit.  $\square$

## 6. THE BREUIL FRAMES

We return to the notation fixed in the introduction, in particular  $p \geq 3$ . Let  $I_a = E\mathfrak{S}_a$  be the kernel of  $\mathfrak{S}_a \rightarrow R_a$ ; note that  $E$  is not a zero divisor in  $\mathfrak{S}_a$ . The Frobenius  $f$  of  $W(k)$  is extended to  $\mathfrak{S}_a$  by  $f(u) = u^p$  and  $f(t_i) = t_i^p$ . For  $x \in I_a$  let  $f_1(x) = f(x/E)$ . Then

$$\mathcal{B}_a = (\mathfrak{S}_a, I_a, R_a, f, f_1)$$

is a frame with  $\pi = f(E)$ . Windows over  $\mathcal{B}_a$  in the sense of Definition 1.2 are (equivalent to) the Breuil windows relative to  $\mathfrak{S}_a \rightarrow R_a$  introduced in [VZ]; see loc.cit., Lemma 1.

The frames  $\mathcal{B}_a$  are related with Dieudonné frames as follows. There is a unique ring homomorphism  $\varkappa : \mathfrak{S} \rightarrow \mathbb{W}(R)$  that lifts the projection  $\mathfrak{S} \rightarrow R$  and commutes with  $f$ . By [VZ], Lemma 2, we have  $\varkappa(f(E)) = pu$  for a unit  $u \in R$ . It is easy to see that  $\varkappa$  induces compatible  $u$ -morphisms of frames  $\varkappa_a : \mathcal{B}_a \rightarrow \mathcal{F}_{R_a}$  lying over the identity of  $R_a$ .

**Theorem 6.1.** *For all  $a \in \mathbb{N} \cup \{\infty\}$  the morphism  $\varkappa_a$  is crystalline.*

The case  $a = 1$  follows from [Z3], and  $a = \infty$  is the main result of [VZ].

*Proof.* For  $a \in \mathbb{N}$  we construct a commutative diagram of frames such that vertical arrows are  $u$ -morphisms and horizontal arrows are 1-morphisms:

$$(6.1) \quad \begin{array}{ccccc} \mathcal{B}_{a+1} & \xrightarrow{\beta_1} & \mathcal{B}' & \xrightarrow{\beta_2} & \mathcal{B}_a \\ \downarrow \varkappa_{a+1} & & \downarrow \varkappa' & & \downarrow \varkappa_a \\ \mathcal{F}_{R_{a+1}} & \xrightarrow{\alpha_1} & \mathcal{F}' & \xrightarrow{\alpha_2} & \mathcal{F}_{R_a} \end{array}$$

The lower line is the factorisation (5.1) of the projection  $\mathcal{F}_{R_{a+1}} \rightarrow \mathcal{F}_{R_a}$  with respect to the trivial divided powers on the kernel  $\mathfrak{b} = p^a R_{a+1}$ . In particular,  $\alpha_2$  is crystalline by Proposition 5.1.

Let  $\mathfrak{a} = u^{ae} \mathfrak{S}_{a+1} = \text{Ker}(\mathfrak{S}_{a+1} \rightarrow \mathfrak{S}_a)$ . We define a frame

$$\mathcal{B}' = (\mathfrak{S}_{a+1}, I', R_a, f, f'_1)$$

where  $I' = I_{a+1} + \mathfrak{a}$ , and where  $f'_1 : I' \rightarrow \mathfrak{S}_{a+1}$  is the unique extension of  $f_1$  with  $f'_1(\mathfrak{a}) = 0$ . This is well-defined because  $f_1 = 0$  on  $\mathfrak{a} \cap I_{a+1} = u^{ae} E \mathfrak{S}_{a+1}$ . Hence the upper line of (6.1) is constructed too. Here  $\beta_2$  is crystalline by Theorem 3.2; the required filtration of  $\mathfrak{a}$  is trivial.

In order that  $\varkappa'$ , necessarily given by  $\varkappa_{a+1}$ , is a  $u$ -morphism of frames, we need that  $f_1(\varkappa_{a+1}(u^{ae}x)) = 0$  for  $x \in \mathfrak{S}_{a+1}$ . Now  $\varkappa_{a+1}(u^{ae})$  is the Teichmüller element  $[u^{ae}]$  and  $\log([u^{ae}]) = [u^{ae}, 0, 0, \dots]$ . Hence  $f_1([u^{ae}]) = 0$  and thus  $f_1(\varkappa_{a+1}(u^{ae}x)) = f_1(\varkappa_{a+1}(u^{ae}))f(\varkappa_{a+1}(x)) = 0$  as required.

It follows that lifts of windows from  $\mathcal{B}_a$  to  $\mathcal{B}_{a+1}$  or from  $\mathcal{F}_{R_a}$  to  $\mathcal{F}_{R_{a+1}}$  are both classified by lifts of the Hodge filtration in the same way. Since  $\varkappa_1$  is crystalline by [Z3], by induction  $\varkappa_a$  is crystalline for all finite  $a$ , so  $\varkappa_\infty$  is crystalline by passing to the projective limit.  $\square$

*Remark 6.2.* Along the same lines one can show directly that  $\varkappa_1$  is crystalline. Namely, for each  $\underline{n} = (n_1, \dots, n_{r+1})$  with  $n_i \in \mathbb{N} \cup \infty$  let

$$S_{\underline{n}} = W(k)[[t_1, \dots, t_{r+1}]]/(t_1^{n_1}, \dots, t_{r+1}^{n_{r+1}})$$

with  $t_i^\infty = 0$ . Let  $I_{\underline{n}} = pS_{\underline{n}}$  and  $R_{\underline{n}} = S_{\underline{n}}/I_{\underline{n}}$ . We have a frame

$$\mathcal{C}_{\underline{n}} = (S_{\underline{n}}, I_{\underline{n}}, R_{\underline{n}}, f, f_1)$$

where  $f(t_i) = t_i^p$  and  $f_1(x) = f(x/p)$  for  $x \in I_{\underline{n}}$ , and compatible morphisms of frames  $\varkappa_{\underline{n}} : \mathcal{C}_{\underline{n}} \rightarrow \mathcal{F}_{R_{\underline{n}}}$ . Mutatis mutandis the proof of Theorem 6.1 shows that these are all crystalline; in the initial case  $\underline{n} = (1, \dots, 1)$  the morphism  $\varkappa_{\underline{n}}$  is an isomorphism. In  $\mathfrak{S}_1$  we have  $f(E) = vp$  for a unit  $v$ , and for  $\underline{n} = (\infty, \dots, \infty, e)$  there is a  $v^{-1}$ -isomorphism of frames  $\mathcal{C}_{\underline{n}} \cong \mathcal{B}_1$  compatible with the respective  $\varkappa$ 's. The assertion follows.

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